6 Conjoint Measurement: A Brief Survey

R. Duncan Luce

6.1 INTRODUCTION

At its most general level, the purpose of conjoint measurement is to understand what sorts of numerical representations exist, if any, for orderings of Cartesian products of sets. The problem is ubiquitous:

1. In physics, one can order pairs consisting of a homogeneous substance and a volume by the mass of that volume of the substance – this can be done with a pan balance and a set of containers without having numerical measures of either mass or volume or, of course, substance. Presumably, if we understand matters correctly, we should end up with numerical measures of mass, volume, and density.
2. In economics, one can order commodity bundles – a listing of amounts of various goods – by preference. A numerical representation corresponding to preference would, on one hand, be a kind of utility measure and, on the other, tell something about how different commodities are aggregated by an individual.
3. In economics, statistics, and psychology, one can order gambles – consequences assigned to chance events – by riskiness and thereby arrive at a numerical scale of risk that shows how the consequences combine with the events to yield a measure of risk.
4. In psychology, one can present one intensity of a pure tone to one ear and a different intensity to the other ear, producing an overall sensation of loudness that is some composite of the loudnesses perceived by each ear separately. Given a subject's ordering by loudness of such pairs, we may study the existence of a numerical scale of loudness and of a law for combining loudnesses between the ears.

The problem is one of uncovering both scales of measurement of the factors and a law for combining these scales to form a composite or conjoint scale that

recovers the qualitative ordering. Often the problem is partially constrained by the existence of already known scales, derived in other ways, that should be related simply to those obtained by conjoint methods. For example, in physical measurement, we anticipate that the conjoint measures of mass and volume should relate simply to the usual measures derived from the theory of extensive measurement, which is based on the existence of an operation of combination that preserves the attribute in question. Such "concatenation" operations are typified by placing two masses together on a pan balance or by abutting two rods to form a new rod.

We shall confine our attention to one particularly simple class of numerical representations. Let A and P be sets and $\succsim$ a binary relation on $A \times P$. We say that the structure $\langle A \times P, \succsim \rangle$ is *decomposable* if and only if there exist functions ϕ_A on A and ϕ_P on P into the real numbers and a function F from the real plane into the reals such that $F(\phi_A, \phi_P)$ represents $\succsim$. To be specific, for all a, b in A and p, q in P,

$$(a, p) \succsim (b, q) \text{ iff } F[\phi_A(a), \phi_P(p)] \geqslant F[\phi_A(b), \phi_P(q)].$$

Another way to describe this is to say that there is a binary operation $\odot$ on the reals such that $\phi_A \odot \phi_P$ represents $\succsim$, where, of course, for all real r and s, the following relation between $\odot$ and F holds:

$$r \odot s = F(r, s).$$

Our attention will be restricted to the decomposable case and the obvious generalization of that concept for three or more factors.

Although this sort of representation seems incredibly general – certainly it appears to cover all of the two-factor cases one runs into in physics – there are simple, and perhaps interesting, cases not covered by it. For example, suppose ϕ_A, ψ_A, and ϕ_P, ψ_P are functions on A and P, respectively, and $\succsim$ is the ordering generated on $A \times P$ by

$$\phi_A + \phi_P + \psi_A \psi_P;$$

then, except for a few special cases, $\langle A \times P, \succsim \rangle$ is not a decomposable structure. Nevertheless, this representation seems interesting because the interaction term that is added to the additive part of a representation is itself multiplicatively independent. The only relevant work (Fishburn, 1975) makes it clear that it will probably be difficult to understand what properties of $\succsim$ lead to such a representation.

When one already has measures of either the two factors A and P or of one of those and of $\succsim$ on $A \times P$ – this is true of examples 1, 2, and 4 above – then one can represent the information given in the problem by means of *indifference curves*. In the physical example, we have measures of mass and of volume, so we identify with each substance the locus of mass–volume pairs that can arise. Each substance will generate its own curve, and under reasonable assumptions these will not intersect. In the psychological example, we have physical measures of the intensities presented to the two ears (usually we use the logarithm of intensities – the decibel scale), and we plot the loci of intensity pairs that are judged to produce equivalent loudness. The problem in these cases is to find a suitable numerical representation of the indifference curves.

The question becomes one of deciding on what we mean by a suitable representation. It appears that the only formal requirement for acceptance of the result as a form of measurement is that the representation be nearly unique. Often the degree of uniqueness is described in terms of the group (or semigroup) of transformations that take a representation involving $\odot$ into another representation based on the same $\odot$. For example, in the usual theory of mass measurement, the concatenation operation is mapped into $+$ and $\succsim$ is mapped into $\geqslant$, and it is shown that only multiplication by a positive constant takes one representation into another representation. This corresponds to a change in units. However, when $\odot$ is more complex than $+$ or $\cdot$, explicitly describing the class of admissible transformations can be difficult. It seems better simply to say how many values of the representation must be specified in order to determine it uniquely. In the case of mass, one value is sufficient. Thus, our interest will be in constructing representations that become unique when their values are specified at one or a few points.

The rest of the paper deals with most of what is known about decomposable representations. First, both historically and in terms of mathematical simplicity, is the additive case: when $\odot$ is $+$. This is the most fully understood case and in many ways serves as an underpinning to more general ones. Second, we look at several results of a nonadditive sort. Third, we examine the problem of relating conjoint structures to concatenation structures and the way in which this provides some better understanding of the interplay of addition and multiplication, which is exploited in dimensional analysis. Finally, we look at the, as yet, small literature concerned with random variable representations of conjoint structures.

My main concern is with the key ideas and the general spirit, and so I shall slight various points of mathematical nicety. The interested reader will find the precise definitions and theorems in the references cited.

6.2 THE ADDITIVE REPRESENTATION

6.2.1 THE INFINITE CASE

Ideally, one would like to know two things: what are necessary and sufficient conditions for $\langle A \times P, \succsim \rangle$ to have an additive representation, and how does one construct the functions ϕ_A and ϕ_P? We cannot yet answer either question fully. When A and P are finite sets, necessary and sufficient conditions are known (see section 6.2.2, below), and the representation, whose uniqueness is very difficult to characterize, involves finding the solutions to a system of linear inequalities. Algorithms for doing this are known and, because of the speed of computers, are

feasible if the sets are not too large. In the infinite case, we know only sufficient conditions for the existence of an additive representation, but we have systematic procedures to approximate ϕ_A and ϕ_P, and they are unique up to specification at two points. We take up the infinite case first.

The axioms are conveniently grouped into three types:

First-order axioms: necessary conditions. These include at least the following three properties, each of which derives immediately from the intended representation by cancellations in the corresponding linear inequalities.

Transitivity: if $(a, p) \succsim (b, q)$ and $(b, q) \succsim (c, r)$, then $(a, p) \succsim (c, r)$.

Independence: if $(a, p) \succsim (b, p)$, then $(a, q) \succsim (b, q)$; if $(a, p) \succsim (a, q)$, then $(b, p) \succsim (b, q)$.

Double cancellation[1]: if $(a, x) \succsim (f, q)$ and $(f, p) \succsim (b, x)$, then $(a, p) \succsim (b, q)$.

First-order axioms: structural conditions. These are not necessary consequences of this representation; rather, they constrain the structures in ways that are thought to be useful. Various requirements of nontrivialness are of this sort. Of more substantive interest are axioms that assume the existence of elements where one wants them. If $\sim$ is defined as above, the strongest form of *solvability* says that one can always construct a complete indifference curve passing through any prescribed point. That is, given a, b in A and p in P, there exists q in P such that $(b, q) \sim (a, p)$. Similarly, given a in A and p, q in P, there exists b in A such that $(b, q) \sim (a, p)$. As this is restrictive in many cases, a weaker form of solvability, called *restricted solvability,* is usually invoked: it says, for the first component, that if there are $\bar{b}$ and b in A such that $(\bar{b}, q) \succsim (a, p) \succsim (b, q)$, then b exists in A such that $(b, q) \sim (a, p)$. A similar statement holds for the second component.

The primary role played by these solvability conditions is to ensure the existence of what amounts to equally spaced elements. The method of finding such a sequence in one coordinate is to balance them off against a pair of elements in the other coordinate. Thus, we say that a sequence a_i in A, where i is in a set I of successive integers, is a *standard sequence* if there are p, q in P such that for all $i, i+1$ in I,

$$(a_i, p) \sim (a_{i+1}, q).$$

We see that if an additive representation exists, then

$$\phi_A(a_i) - \phi_A(a_{i+1}) = \phi_P(q) - \phi_P(p),$$

and so the successive intervals of the standard sequences are, indeed, equal. Without solvability, there is no reason for any standard sequences to exist. With solvability,

[1] If $(a, p) \sim (b, q)$ is defined to mean $(a, p) \succsim (b, q)$ and $(b, q) \succsim (a, p)$, then double cancellation for $\sim$ is called the Thomson condition.

even restricted solvability, some do exist, and they play a vital role in the construction of the representation.

Second-order axioms. The third kind of axiom is again a necessary condition, but it is a second-order axiom in the sense of logic. The one usually invoked is called an Archimedean axiom.[1] It simply states that any standard sequence that is bounded from above and below is finite. Such second-order axioms are infuriating to the empirical scientist because they cannot be tested directly. For this reason, it is reassuring to know that the Archimedean axiom is dispensable, provided one is willing to accept representations into the nonstandard reals (Narens, 1974a; Skala, 1975). Put another way, $(a, p) \succ (b, q)$ maps into $\phi_A(a) + \phi_P(p) \geqslant \phi_A(b) + \phi_P(q)$ ($\succ$ goes into $\geqslant$, rather than into $>$).

The major theorems in the infinite case invoke a mix of first-order necessary and structural axioms, together with the Archimedean axiom, to prove the existence of functions ϕ_A on A and ϕ_P on P such that $\phi_A + \phi_P$ is order-preserving. Moreover, if ϕ'_A, ϕ'_P is another representation, then there are constants $\alpha > 0$, β_A, and β_P such that $\phi'_A = \alpha\phi_A + \beta_A$, $\phi'_P = \alpha\phi_P + \beta_P$.

There is a distinct trade-off between the strength of the necessary axioms one needs to invoke and the strength of the structural conditions. For example, Luce and Tukey (1964) and Debreu (1960), who used strong topological assumptions, invoked weak ordering (transitivity plus connectedness of $\succsim$), double cancellation, nontrivialness, and solvability. Later, following work of Holman (1971) and Luce (1966), Krantz *et al.* (1971) used weak ordering, independence, Thomson condition, non-trivialness, and restricted solvability. The trade-off involves weakening solvability to restricted solvability and compensating by replacing double cancellation with the weaker Thomson condition together with the property of independence. Both systems also invoke the Archimedean property.

The method of proof is of some interest. Independence means that $\succsim$ induces a unique weak order on each component; e.g., $\succsim_A$ is defined by $a \succsim_A b$ iff $(a, p) \succsim (b, p)$ for some p in P, and $\succsim_P$ is defined similarly. Fix a_0 and p_0. By solvability, define the function π by $(a_0, \pi(a)) \sim (a, p_0)$, and by solvability define the operation $\circ_A$ on A as the solution to

$$(a \circ_A b, p_0) \sim [a, \pi(b)].$$

In this construction, a_0 acts like a zero element because $\pi(a_0) \sim_P p_0$ and

$$(a \circ_A a_0, p_0) \sim (a, \pi(a_0)) \sim (a, p_0),$$

whence $a \circ_A a_0 \sim_A a$. It turns out that the set of elements A_0 of A that are $\succ_A a_0$ and the restriction of $\circ_A$ and $\succsim_A$ to that set behave mathematically just like

[1] In some axiomatizations, the Archimedean axiom and the solvability condition are combined into a somewhat stronger topological axiom called completeness. See Ramsey (1975) for a defense of this and Narens and Luce (1975) for an objection to it. In other axiomatizations the existence of a countable order-dense subset is postulated.

length or mass, and the theory of extensive measurement (see Krantz *et al.*, 1971, Chapters 2 and 3) ensures the existence of ϕ_A that is additive over $\circ_A$. This function is easily extended to the elements $\precsim a_0$. The methods for approximating it in terms of a standard sequence are well known. In particular, if a standard sequence a_i, beginning with a_0, is constructed relative to p_0 and $p_1 (\succ p_0)$, then we can choose ϕ_A so that $\phi_A(a_i) = i$. For any $a \succ a_0$, and any n, find $i(n)$ such that

$$a_{i(n)} \succ na \succsim a_{i(n)-1},$$

which is possible according to the Archimedean axiom. Thus,

$$\frac{i(n)}{n} > \phi_A(n) \geqslant \frac{i(n) - 1}{n},$$

and so, to within an error of $1/n$, $\phi_A(a) = i(n)/n$. This means that the construction of the conjoint representation is feasible via standard sequences.

The function ϕ_P can be constructed similarly; however, it is easier merely to define it as

$$\phi_P(p) = \phi_A \pi^{-1}(p).$$

Thus the problem is reduced to proving that $\phi_A + \phi_P$ is order-preserving.

Additive conjoint measurement on three or more factors is a bit simpler. Independence is generalized to mean that the ordering induced on any set of factors is independent of the common value selected for the complementary factors. No analog of double cancellation or Thomson condition is needed because, in the presence of the other axioms, these conditions can be derived for any pair of factors. So, when there are three or more factors, an additive representation exists if the following conditions hold: weak ordering, independence, restricted solvability, nontrivialness, and an Archimedean property.

Another approach, not involving solvability conditions but invoking many cancellation properties, which extends techniques from the finite case to the infinite one, can be found in Jaffray (1974).

6.2.2 THE FINITE CASE

Data developed using standard sequence techniques are rather more special than an ordering of an arbitrary finite $A \times P$, which arises when one runs a straightforward factorial design. In the latter case, no solvability properties whatever will be satisfied, and so all one has to work with are the first-order necessary conditions. These include the ones we have listed and any others that can be derived from linear inequalities by canceling common terms. One might hope that a finite set of such inequalities would suffice, but Scott and Suppes (1958) proved that to be impossible. The number of inequalities necessary and sufficient for an additive representation increases with the number of elements in $A \times P$. Scott (1964) and Tversky (1964) independently devised a compact way of formulating these necessary and sufficient conditions (see Krantz *et al.*, 1971, Chapter 9). This type of condition has been generalized to countable and noncountable situations by Jaffray (1974).

In practice, one simply writes down for each data inequality the corresponding numerical linear inequality and then searches for a solution to the resulting system. Computer programs for doing this have been developed by Tversky and Zivian (1966) and Young (1973), among others.

Narens (1974b) raised the question of when a nested collection of finite additive conjoint structures approaches a countable additive conjoint structure whose representation is unique up to interval scales. To get at this, let $\succsim$ be an independent weak ordering of $A \times P$. On A, define

$$a - b \succ_A c - d \text{ iff there exist } p, q \text{ in } P \text{ such that } (a, p) \succ (b, q) \text{ and } (c, q) \succ (d, p).$$

In terms of this notion, for $b \succ_A a$ we say c, d in A form a *trisplit* if

$$b \succ_A d \succ_A c \succ_A a,\ b - c \succ_A c - a,\ d - a \succ_A b - d,\ b - c \succ_A d - c,$$
$$\text{and } c - a \succ_A d - c.$$

If for each pair of elements from each component there is a trisplit, then we say the conjoint structure is *trisplittable*. Narens shows that if each member of the nested set is trisplittable, then the above convergence obtains.

6.2.3 FUNCTIONAL MEASUREMENT

Anderson and his associates have published extensively on functional measurement, which is, in many ways, closely related to conjoint measurement (see Anderson, 1970, 1971, and 1974, for surveys and bibliographies). There are two key features of Anderson's work. First, the data take the form of numbers, usually arising from some sort of rating or category method: for example, if the stimulus (a, p) consists of sound intensities to the two ears, the subject provides a loudness rating that fits into, say, one of seven categories. Second, the data (or, in some cases, some transformation of them) are assumed to satisfy some explicit representation. One case is the additive representation just discussed, but much more important in Anderson's work have been representations of weighted averages. If I understand him correctly, these representations are axiomatized by the bisymmetric operations of section 6.4.1, below.

Various techniques, closely allied to analysis of variance methods, are used to obtain the scales from the category ratings. These methods have been applied to a wide range of situations, from psychophysics to impressions of personality obtained from verbal descriptions.

Anderson makes much of the internal consistency he finds when these representations are coupled with category scaling, and he criticizes (Anderson, 1970) another method initiated by Stevens (1957, 1975) and widely used in psychophysics. Stevens' magnitude estimation differs from the category methods in that

the range of possible numerical response is not limited and the subject is asked to use the numbers so that they reflect the subjective ratios of stimuli. On the face of it, one would not expect the additive or averaging models to be appropriate to these instructions, but certain multiplicative representations might be. Because magnitude methods are very easy to use and, at least in psychophysics, are widely useful and their detailed properties are becoming better understood (Green and Luce, 1974; Marks, 1974; Moskowitz *et al.*, 1974; Stevens, 1971, 1975), they probably should be given serious consideration by students of factorial situations.

Both the category and magnitude methods have, for factorial designs, the advantage over the ordering methods in that they provide a numerical scale without requiring the solution of systems of linear inequalities. All these methods, of course, have the disadvantage, compared with studying axioms individually, of not localizing the difficulty when the model fails to fit the data.

A great deal of controversy exists over the relationship between functional and conjoint measurement – see, for example, the criticism of Anderson (1971) by Hodges (1973) and Schönemann *et al.* (1973), with a reply by Anderson (1973). Roughly, the lines are drawn as follows. Measurement theorists point out that no qualitative representation theorems are proved in the functional measurement literature, to which the reply is "What good are such theorems?" The answer is, first, that direct tests of axioms appear to be more revealing of the failure of a representation than is fitting it to factorial data, and, second, that the proofs of the theorems suggest ways to construct representations in nonfactorial situations. Anderson (1974) points out that his empirical methods go far beyond anything found in the conjoint measurement literature. They suggest a substantive hypothesis that, if true, is important; this hypothesis holds that particular data collection procedures yield the representation directly without further transformation and that techniques of the analysis of variance can be employed to cope with error. Moreover, he and his colleagues have collected far more data than all the measurement theorists put together, and these data do not support the additive representation that has been so much the focus of conjoint measurement. The measurement theorists have had little to say in reply, although they have informally criticized specific studies (complaining, for example, that the dynamic ranges used in the psychophysical studies are too narrow to test his methods rigorously). In my view, the methods are largely complementary, not competitive.

6.2.4 UNIFORM SYSTEMS AND INDIFFERENCE CURVES

In the special, but important, case when there are numerical measures (often physical, but not always) on A and P that agree with $\succsim_A$ and $\succsim_P$, respectively, the problem can be recast in terms of indifference curves in the plane. In fact, the additive case is equivalent to finding transformations of the two given scales so that the indifference curves become straight lines with slopes -1. When one has all possible indifference curves in the plane, the theory of webs provides the solution (Blaschke and Bol, 1938; Aczél *et al.*, 1960; Havel, 1966; Radó, 1960, 1965). However, in practice, one usually has only a finite amount of information about each of a finite number of curves. By reasonable interpolation, one can replace this situation by a situation in which one knows a finite number of indifference curves completely. Note that this situation is not exactly like the infinite one, where one develops standard sequences, and it is certainly different from the finite factorial one. Levine (1970, 1972) has studied such systems and generalizations of them; some of his results are summarized by Krantz *et al.* (1971, section 6.7). Roughly, one constructs from any two given indifference curves F and G another curve of the form $F^{-1}G$, and these curves can be transformed into the additive form if and only if none of the original curves nor any generated recursively by forming $F^{-1}G$ intersect.

Levine (1975a, b) has been developing computer methods, based heavily on the group theoretic character of his theorems, to make the search for additive (and other) representations practical. He is applying these techniques to latent trace models for test theory.

6.3 NONADDITIVE REPRESENTATIONS

Since even the additive case is far from fully understood, we can anticipate only partial results in nonadditive cases. Again, we must distinguish between the infinite and finite situations. As the results in the finite case are quite abstract (see section 9.5 of Krantz *et al.*, 1971), and since they have not, to my knowledge, been applied, I will not summarize them. So we deal with the infinite case.

Recall that a structure of the form $\langle \times_{i=1}^{n} A_i, \succsim \rangle$ is *decomposable* if and only if there are real-valued mappings ϕ_i on A_i and a real-valued function F of n real variables such that

$$(a_1, \ldots, a_n) \succsim (b_1, \ldots, b_n) \text{ iff } F[\phi_1(a_1), \ldots, \phi_n(a_n)] \succsim F[\phi_1(b_1), \ldots, \phi_n(b_n)] .$$

It is *monotonically decomposable* if, in addition, F is strictly monotonic in each of its arguments. Krantz *et al.* (1971, section 7.2) give necessary and sufficient conditions for the existence of such a monotonically decomposable representation: $\succsim$ must be a weak ordering, the equivalence classes of $A = \times_{i=1}^{n} A_i$ under $\succsim$ must have a countable order-dense subset (i.e., a countable set B such that between any two distinct elements of A there is an element from B), and each A_i must be independent of the remaining components in the sense that for each i

$$(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n) \succsim (a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)$$

if and only if

$$(b_1, \ldots, b_{i-1}, a, b_{i+1}, \ldots, b_n) \succsim (b_1, \ldots, b_{i-1}, b, b_{i+1}, \ldots, b_n).$$

Expressed verbally, the ordering established on A_i is independent of the fixed choices on the remaining components.

This result is less than satisfactory in two ways. It involves an awkward second-order axiom, namely, the existence of a countable order-dense subset, and it provides no insight into constructing the representation. For the case of two components, Narens and Luce (1975) drop the countability requirement, and they show how to reduce the construction of the representation to that of a non-associative concatenation structure. Moreover, they work with a local (not connected) ordering, which is sometimes useful. In the special case of a weak ordering, their axioms are essentially those of the additive case, minus the Thomson condition plus density: weak ordering, independence, nontrivialness, solvability, density, and an Archimedean property. Using the same definition of $\circ_A$ as in the additive case, they show that $\langle A, \succsim_A, \circ_A \rangle$ is a positive concatenation structure that is associative only if the Thomson condition holds. Thus, the problem of constructing the representation is reduced to that of constructing one for the operation $\circ_A$. This is by no means generally understood.

More specialized results of Krantz (1968), Krantz and Tversky (1971) (summarized in sections 7.3–7.4 of Krantz *et al.*, 1971), and Falmagne (1973) hold for simple polynomial representations. A simple polynomial is defined inductively as one for which the variables can be partitioned into two sets such that the given polynomial is either the sum or the product of simple polynomials on the two sets. For example, $(\phi_1 + \phi_2)\phi_3\phi_4$ is simple, but $\phi_1\phi_2 + \phi_2\phi_3 + \phi_1\phi_3$ is not. One can, in the presence of solvability conditions, work out necessary properties that permit distinction among these cases. These properties, which are too complex to state here, permit one to search for additivity, $\phi_1 + \phi_2$; multiplicativity, $\phi_1\phi_2$; and various types of cancellation properties that arise from the distributive property $\phi_1(\phi_2 + \phi_3) = \phi_1\phi_2 + \phi_1\phi_3$. The models have been worked out in detail for $n = 3$, where the four simple polynomials are

$$\phi_1 + \phi_2 + \phi_3, \phi_1\phi_2\phi_3, \phi_1(\phi_2 + \phi_3), \quad \text{and} \quad \phi_1\phi_2 + \phi_3,$$

plus permutations on the indices of the last two. Some applications of these methods are described below.

A general discussion of polynomial measurement in the finite case is given by Tversky (1967a) (see section 9.5 of Krantz *et al.*, 1971); Richter (1975) has resolved a conjecture on the conditions under which systems of polynomial inequalities have a solution.

6.4 RELATIONS TO OTHER FORMS OF MEASUREMENT

Conjoint measurement has proved more useful at a theoretical level than at an empirical one. It is not that we lack empirical applications – see section 6.5 – but that applications of conjoint measurement to theoretical problems have provided deeper insight than have the empirical applications. Three applications are described in this section, and another has been described in the preceding section – namely, a fairly general case where conjoint structures reduce to nonassociative positive concatenation structures; our first example below is its converse.

6.4.1 CONCATENATION STRUCTURES

Classical physics exhibits two quite different kinds of binary operations – *extensive* ones, such as juxtaposition of rods for length or set theoretic union for mass; and *intensive* ones, such as temperature and density. The former are *positive* in the sense that $a \circ b \succ a, b$ and also associative, whereas the latter are *intern* in the sense that if $a \succ b$, then $a \succ a \circ b \succ b$. If we drop all three special properties, we have what are called *concatenation structures*.

Assume that the structure $\langle A, \succsim, \circ \rangle$ satisfies the following properties:

$\succsim$ is a nontrivial weak order; i.e., for some a, b in A, $a \succ b$.

Monotonicity: $a \succsim b$ iff $a \circ c \succsim b \circ c$ iff $c \circ a \succsim c \circ b$.

Restricted solvability: if $\bar{b} \circ c \succsim a \succsim b \circ c$, then there are b in A such that $b \circ c \sim a$ (the parallel statement on the right is also true).

Archimedean property: let standard sequence a_i satisfy $a_i \circ p \sim a_{i+1} \circ q$ or $p \circ a_i \sim q \circ a_{i+1}$ for some p and q, then every bounded standard sequence is finite.

If $\succsim'$ on $A \times A$ is defined by

$$(a, b) \succsim' (c, d) \text{ iff } a \circ b \succsim c \circ d,$$

then it can be shown (see Krantz *et al.*, 1971, section 6.101) that the conjoint structure $\langle A \times A, \succsim' \rangle$ satisfies the requirements for weak ordering, independence, restricted solvability, the Archimedean property, and nontrivialness. Moreover, if we add the important property of bisymmetry,

$$(a \circ b) \circ (c \circ d) \sim (a \circ c) \circ (b \circ d),$$

then $\succsim'$ satisfies double cancellation. In the latter case, we can use the additive conjoint representation and prove there is an order-preserving representation ϕ and constants $\mu \succ 0$, $\nu \succ 0$, λ such that

$$\phi(a \circ b) = \mu\phi(a) + \nu\phi(b) + \lambda.$$

Additional properties on $\circ$ place restrictions on the constants:

If $a \circ a \sim a$, as in the intensive case, $\mu + \nu = 1$ and $\lambda = 0$.

If $\circ$ is commutative, i.e., $a \circ b \sim b \circ a$, then $\mu = \nu$.

If $\circ$ is associative and commutative, as in the extensive case, $\mu = \nu = 1$ and, with no loss of generality, $\lambda = 0$.

Narens and Luce (1975) have shown that there is a complementary relation between one general class of intensive structures and nonassociative, positive concatenation structures with half elements.

6.4.2 CONDITIONAL EXPECTED UTILITY

In an attempt to overcome some of the criticisms of Savage's (1954) important axiomatization of subjective expected utility, Luce and Krantz (1971) (see Krantz *et al.*, 1971, Chapter 8) axiomatized a notion of conditional decisions. One may think of their formulation as based on an algebra $\mathfrak{E}$ of events, a family $\mathfrak{D}$ of conditional decisions that can be written f_A, where A is in $\mathfrak{E}$ (this suggests that f_A is a function from A into some set of consequences, which is one interpretation of the model), and an ordering $\succsim$ of $\mathfrak{D}$. One of their key assumptions, and one that has been strongly criticized by Balch (1974), Balch and Fishburn (1974), and Fishburn (1974) and defended by Krantz and Luce (1974), is that $\mathfrak{D}$ is closed under unions of decisions on disjoint events and under restrictions to nonempty subevents. The axioms, too complex to restate here, are sufficient to show that a real-valued function u exists on $\mathfrak{D}$ that is order-preserving and a probability measure P exists on $\mathfrak{E}$ such that for A, B in $\mathfrak{E}$ with $A \cap B = \phi$ and f_A, g_B in $\mathfrak{D}$,

$$u(f_A \cup g_B) = u(f_A)\, P(A|A \cup B) + u(g_B) P(B|A \cup B).$$

This is the conditional expected utility property.

No attempt is made here to outline the proof, but it involves looking at all the decisions on triples of mutually disjoint events, showing that these are additive conjoint structures, and using the uniqueness theorem for such structures to introduce the probability measures. To the extent that such structures are of interest in decision making (and I think there are good reasons to believe they are considerably more satisfactory than Savage's system), conjoint measurement has been put to important use.

6.4.3 ALGEBRA OF PHYSICAL QUANTITIES

Physical measures exhibit two quite different numerical structures; some involve an operation that satisfies the axioms of extensive (or more general) concatenation measurement, and some triples of measures are related by equations of the form

$$z = x^{\alpha} y^{\beta},$$

where some or all of x, y, and z are extensive measures. In the latter case, it is clear that the ordering induced by z on the structure of (x, y) pairs must satisfy the axioms of additive conjoint measurement. So, again, we see that this algebra must play a role in a qualitative development of the measurement underpinnings of physical measures. The main problems in building such a theory are, first, to formulate the qualitative interlock between the extensive and conjoint structures and, second, to characterize the exponents α and β. Krantz *et al.* (1971, Chapter 10) have made an attempt to do this, based primarily on the work of Luce (1965). An improved version is provided by Narens and Luce (1975), and it is outlined here.

Basically, there are two cases to be considered. Suppose that $\langle A \times P, \succsim \rangle$ is a conjoint structure satisfying independence and that there is either an operation $\circ_A$ on A (or $\circ_P$ on P) or an operation $\circ$ on $A \times P$. In the former case we assume the following distribution condition:

$$\text{if } (a, p) \sim (c, q) \text{ and } (b, p) \sim (d, q), \text{ then } (a \circ b, p) \sim (c \circ d, q).$$

In the latter case, distribution takes the form

$$(a, p) \circ (b, p) \sim (c, p) \text{ iff } (a, q) \circ (b, q) \sim (c, q).$$

This permits us to define $\circ_A$ by

$$a \circ_A b \;=\; c \text{ if for some } p, \text{ hence for any } p, (a, p) \circ (b, p) \sim (c, p),$$

and this operation satisfies the first condition. What Narens and Luce prove is that, under solvability conditions, if $\circ_A$ is an extensive operation with an additive representation ϕ_A, then there is a scale ϕ_P' on P such that $\phi_A \phi_P'$ is order-preserving. Thus, distributivity coupled with extensiveness forces the conjoint structure to be additive. If, in addition, $\circ_P$ exists and is extensive with an additive representation ϕ_P, then there are constants α, β such that $\phi_A^\alpha \phi_P^\beta$ is order-preserving.

Because of the uniqueness of additive conjoint measurement, only the value of α/β is of significance. To characterize its value, one need only state the exchange relation between concatenations on the two factors as follows: there are positive integers m and n such that for all a in A and p in P, either

$$(2^m a, p) \sim (a, 2^n p)$$

or

$$(2^m a, 2^n p) \sim (a, p).$$

Under these conditions, $|\alpha/\beta| = n/m$. Such statements are easily derived from the representation.

In addition to laws of exchange, one must also consider cases where there is an operation $\circ$ on $A \times P$ and $\circ_A$ on A. In such cases, the corresponding laws, called laws of similitude, take the form either

$$2^m (a, p) \sim (2^n a, p)$$

or

$$2^m (2^n a, p) \sim (a, p).$$

Krantz *et al.* (1971, section 10.9) show how a family of physical attributes – some of which are extensive and some triples of which are related, as above, either by laws of exchange or similitude – can be represented, in essence, as a multiplicative vector space with a finite basis of extensive quantities. This is the model of physical measures usually assumed in dimensional analysis. Thus, this theory appears to serve as the qualitative basis for physical measurement – at least in the classical case.

But there is still at least one vexing problem: the interplay of measures in relativistic and quantum physics. The most striking exception to the distribution property is provided by relativistic velocity. Let $D = V \times T$ denote distances formed by pairing velocities with times – all qualitative. Let $\succsim$ be the usual ordering of distances. Let $\circ_D$, $\circ_V$, and $\circ_T$ be the usual concatenations of distance, of velocity (frames of references), and of times. And let ϕ_D, ϕ_V, ϕ_T be the usual numerical measures of distance, velocity, and time. Then, even though $\langle V, \succsim_V, \circ_V \rangle$ is an extensive structure, the representation is:

$$\begin{aligned}
\phi_D(v, t) &= \phi_V(v)\, \phi_T(t) \\
\phi_D[(v, t) \circ_D (v', t')] &= \phi_D(v, t) + \phi_D(v', t') \\
\phi_T(t \circ_T t') &= \phi_T(t) + \phi_T(t') \\
\phi_V(v \circ_V v') &= \frac{\phi_V(v) + \phi_V(v')}{1 + \dfrac{\phi_V(v)\, \phi_V(v')}{\phi_V(c)^2}}
\end{aligned}$$

where c denotes the velocity of light moving in a vacuum. In this case, $\circ_V$ does not satisfy the distributivity condition.

Although the velocity formula has been given qualitative expression by Luce and Narens (In press), it is not really satisfactory for the construction of the algebra of physical quantities. An appropriate analogue to the distributive property is needed.

In summary, it is evident that the concepts of conjoint measurement are essential to a qualitative understanding of the algebra of physical measurement. The difficult problem that remains is to establish the qualitative connection between the extensive and conjoint structures that fail to satisfy distributivity. It is worth noting that, when confronted with this problem, physicists retain the multiplicative representation of the conjoint structure and abandon the additive representation of the extensive one. This is not exactly what one would have anticipated from the emphasis placed by philosophers of physics (e.g., Campbell, 1920; Nagel, 1961) on the additive representation of extensive structures.

6.4.4 MEANINGFULNESS AND DIMENSIONAL INVARIANCE

Dimensional analysis works when one both knows all of the relevant variables and assumes that the law relating them is dimensionally invariant. Roughly, this means that changes in the units of measurement do not alter the mathematical form of the law. More exactly, it means that the law is some unknown function of one or more products of powers of subsets of the variables, where each of the products is dimensionless. The question has long been asked why physical and other scientific laws should exhibit this property (see Chapter 10 of Krantz *et al.*, 1971, for a detailed discussion).

Apparently independent of that discussion, another one in the measurement literature has concerned which statements, framed entirely in terms of a single dimension, can be considered meaningful. For example, it is agreed that it is meaningful to say that one mass is ten times as heavy as another, but that it is not meaningful to say that today's temperature is 10% less than yesterday's. The consensus (see Pfanzagl, 1971) is that a statement is meaningful if and only if it is invariant under the group of transformations describing the uniqueness of the scale. Clearly, this criterion is the natural analogue of dimensional invariance when there is only one dimension.

In both cases one can raise the question of what in the underlying qualitative structure corresponds to a meaningful statement or to a dimensionally invariant one. Recently (Luce, 1976), I have shown that there is a simple answer. Intuitively, a relation in a qualitative relational structure is meaningful if it can be expressed in terms of the relations that define the structure. This is given formal meaning as follows: By an automorphism of the structure one means any one-to-one transformation of the elements that leaves the defining relations invariant. Another relation is then said to be meaningful within a structure if and only if it does not further reduce the set of automorphisms, i.e., it is also invariant under the automorphisms of the structure. It is easy to see that this corresponds exactly to the usual definition of meaningfulness in terms of scales of measurement; it is somewhat more interesting to show in the case of the construction underlying the algebra of physical quantities how it corresponds exactly to dimensional invariance.

6.5 EMPIRICAL APPLICATIONS

It is widely felt that there have been fewer applications of conjoint measurement to date than might be expected from the interest in the theory. There are probably many reasons: difficulty in understanding how the theory and representations relate, doubts about what sort of design is best suited to testing the model and what sort is suited to constructing representations, and lack of satisfactory statistical procedures. Psychologists, for example, have been slow in fully understanding what sorts of data are needed to reject simple algebraic representations. They have long recognized that in 2×2 factor designs "crossed" data – a violation of independence – reject an additive representation. Indeed, as we have seen, a failure of simple factor independence rejects any monotonically decomposable representation. Relatively few understand that to test additivity one should look carefully into double cancellation, which means at a minimum a 3×3 design, or into multiple-factor independence when there are more than two factors. Similarly, the debates between Hull (1952) and Spence (1956) over the formulation of habit strength models amount to choices, based on three factors, between two of the four simple polynomials. It is interesting that in all the years of empirical research on that problem, it appears that no one conducted an experiment adequate to make the choice. One should keep in mind examples such as these, and their waste of effort and resources, when dismissing fundamental measurement theory as an arcane subject of no empirical value.

Examples will be presented here of various empirical approaches that rest on additive conjoint techniques, either directly or indirectly, and on techniques for the simple polynomials. Basically, four approaches have been taken to constructing the representation: rescaling of a numerical function, construction of a function using standard sequence techniques, scaling using factorial methods, and testing of axioms. These three constructions are treated in the next section, and the testing of axioms is treated in section 6.5.2.

6.5.1 CONSTRUCTION OF AN ADDITIVE REPRESENTATION

Scheffé (1959) (see discussion in Krantz *et al.*, 1971, section 6.5.3; and Aczél, 1965) provided a mathematical solution to the following problem: given a numerical scale ϕ on $A \times P$, when is there f, ϕ_A, and ϕ_P such that

$$f\phi = \phi_A + \phi_P?$$

Explicit expressions for f, ϕ_A, and ϕ_P are known. This is true whenever either f is given as an explicit mathematical function or it can be approximated numerically. An example of the former was discussed by Krantz *et al.* (1971, section 6.4.2): Campbell and Masterson (1969) had fit factorial data with a function that, when appropriately transformed, yielded an additive representation.

In principle, the same techniques can be used on data obtained by category or magnitude methods, although in practice (e.g., Anderson, 1974) it seems to be more usual to try to fit the data directly to an additive or averaging model. For example, Feldman and Baird (1971) performed magnitude estimation first on loudness and on brightness separately, finding the usual power functions of physical intensity, and then on the two jointly. They attempted to fit the resulting responses by a geometric mean and by an averaging model based on the functions for the two modalities separately, and the averaging model fit reasonably well.

Although standard sequences are the major theoretical device in constructing additive representations, it seems that only one psychological study has employed it. Levelt *et al.* (1972) used the technique to study loudness summation over the two ears. The stimuli were pairs of intensities of a 1000-Hz tone, with different intensities directed at each ear. Two such stimuli would be presented with only a short time between them, and the subject was asked either to order them by loudness or to modify one of the intensities until a loudness match was achieved. It was found that an additive representation seemed adequate and that the functions ϕ_L and ϕ_R (for the left and right ears) were approximately power functions of physical intensity with exponents comparable to those found using magnitude methods. This result is both satisfying and disquieting. It is satisfying because the growth of loudness is comparable to that found by other methods and because additivity across the ears is so simple. It is disquieting because there are other reasons, among them physiological evidence for crossovers between the ears, to doubt the additivity of loudness. This is treated further in section 6.7.

Perhaps the most congenial approach to conjoint measurement for many social scientists, especially those heavily influenced by analysis-of-variance designs, is the factorial approach. For example, Tversky (1967b) used the method to study the expected utility model in which subjects chose between gambles in which a desired consequence occurred with some probability and nothing occurred with the complementary probability. If the utility of nothing is assumed to be zero, then if c is the consequence and A is the chance event, the subjective expected utility hypothesis (SEU) becomes $u(c)\ P(A)$, which is an additive (under a logarithmic transformation) conjoint representation. Tversky's data, derived from work with prisoners, involved cigarettes and candy as consequences. Additive solutions, using the Tversky and Zivian (1966) program, were found to fit the data well. (For details, see Tversky, 1967b, or Krantz *et al.*, 1971, section 9.4.2.)

The most disturbing aspect of these data, from the point of view of an expected utility theorist, is that if one demands that the probability measure be additive in the sense that $P(A) + P(\bar{A}) = 1$, then it is impossible to demand also that the utility functions obtained from two different procedures, one involving only gambles and the other pure consequences, be the same.

Since 1971, P.E. Green and his associates (Green and Rao, 1971; Green, 1974; Green and Devita, 1974; and Green and Wind, 1975) have been advocating and illustrating the use of the factorial conjoint measurement in various marketing contexts. It remains to be seen whether it will prove sufficiently useful to warrant general adoption.

It is perhaps worth noting here that the techniques of Anderson and his students (for surveys, see Anderson, 1970, 1971, 1974) can be viewed as a program of fitting additive and averaging models to factorial data. The major differences from the techniques so far discussed appear to be that Anderson tends to treat the numerical responses as the scale to be used in testing a representation, that he uses analysis-of-variance techniques on these numbers to decide on the adequacy of the model, and that he usually uses averaging models.

6.5.2 TESTING AXIOMS

Although one may sometimes want or need the numerical representation, the scientific interest is often not in the resulting numbers but in whether we know the relevant independent factors that underlie the subject's behavior. In such cases, it is probably wiser to design studies directly aimed at testing particular axioms, such as independence, double cancellation, or one or another of the distributivity axioms. Krantz (1972, 1974) makes a strong case for this approach, giving a number of illustrations.

We consider studies that have focused on the independence property (which, it will be recalled, implies double cancellation when three or more factors are involved and solvability is satisfied). There is also a literature on transitivity, which we shall not go into here. Perhaps the best-studied example of independence is the property known as the extended sure-thing principle of expected utility theory: if (a, A, b) denotes a gamble in which a is the consequence if event A occurs and b is the consequence if $\bar{A}$ occurs, then the extended sure-thing principle (extended because a may itself be a gamble) is

$$(a, A, b) \succsim (a', A, b) \text{ iff } (a, A, b') \succsim (a', A, b').$$

Ellsberg (1961) focused attention on this property by discussing instances in which reasonable people feel they would violate it. MacCrimmon (1968) reported about 25 percent failure of the property among middle-rank business executives confronted with hypothetical business problems. Becker and Brownson (1974), using graduate students of business, found a violation rate of nearly 50 percent. MacCrimmon and Larsson (In press) give a careful analysis of the problem and report an extensive empirical study in which, again, a substantial proportion of the subjects violate the extended sure-thing principle. It should be realized how sweeping this conclusion is: it not only rejects the SEU model but also invalidates any form of monotonic decomposition that says preferences can be expressed in the form

$$F[\phi(a, A), \phi(b, \bar{A})],$$

where F is strictly increasing in each argument.

The full significance of this for the study of choices under uncertainty seems not to have been fully appreciated. For example, a number of studies attacking SEU (Slovic and Lichtenstein, 1968; Payne, 1973a, b; Payne and Braunstein, 1971) have proposed various alternative models that are decomposable and, hence, are inconsistent with the empirical results cited above.

C.H. Coombs and his students have taken the failures of SEU seriously and have proposed that our preferences for uncertain situations are heavily influenced by a concept of risk. In an attempt to gain some understanding of how various aspects of a gamble affect risk, they have manipulated three factors of gambles and attempted to decide among the four simple polynomial models using the procedures of Krantz and Tversky (1971). Coombs and Huang (1970) showed that only the distributive model was supported. Later, however, Coombs and Bowen (1971), using closely related gambles in which the odds were changed without varying either the expected value or variance, showed that risk varied with the odds. This rejected not only an axiomatization of risk published by Pollatsek and Tversky (1970) but also the distributive model. The cause of this inconsistency is not known.

Tversky and Krantz (1969) did a three-factor study of schematic faces that were varied as follows: long versus wide faces, open versus solid eyes, and straight versus curved mouths. Subjects judged comparative similarities of pairs. Tests of independence were well supported in this case.

A body of literature, typified by the work of Phillips and Edwards (1966) and Edwards (1968), has focused on how well Bayes' theorem describes human probabilistic information processing. If we let Ω_0 denote the prior odds of two hypotheses, H_1 and H_2; Ω_n the posterior odds after observing n sources of information; and L the likelihood ratio of these data having arisen under the two hypotheses, we have from Bayes' theorem the additive representation

$$\log \Omega_n = \log L + \log \Omega_0 .$$

Wallsten (1972) pointed out that in order to understand how the information is being assimilated, it may be useful to regard this as a conjoint measurement problem in the subjects' responses. He reported an experiment in which the probability of the data conditional on each hypothesis, $P(D|H_i)$, and the number n of independent observations were varied. Using the procedures of assessing three-factor simple polynomial models outlined by Krantz and Tversky (1971), the distributive model

$$\phi_3(n)\,[\phi_1(D|H_1) - \phi_2(D|H_2)]$$

was sustained for 8 of 12 subjects. A number of substantive interpretations are made from the calculated functions. Wallsten [In press (a, b)] and Wallsten and Sapp (In press) have followed up this work.

6.6 ERROR

In any attempt either to test or to fit a measurement model to data, a major difficulty is error. Everyone is confident that there is some element of inconsistency in subjects' responses. We know that if we embed a choice in a long series of choices, we do not necessarily get the same response each time it is presented. Whether this is due to fatigue, to changes in attitude resulting from previous choices, or to other sources of variability, we cannot be sure. But whatever the causes, it is clearly inappropriate to demand exact fits of the model to data or to reject an axiom every time an apparent failure occurs.

Although we have been well aware of these difficulties from the first tests of SEU – e.g., those of Mosteller and Nogee (1951) – it is surprising how little has been done to rectify them. Some probabilistic work on transitivity has been done, and some probabilistic choice models have been developed, but within the context of either concatenation measurement structures or conjoint ones, little has been done. The most significant breakthrough is the work of Falmagne (1976); it was motivated by the study of Levelt *et al.* (1972) and the difficulty they had in taking into account the statistical nature of the data.

Falmagne assumes that when a subject is asked to solve an equation of the form

$$(a, p) \sim (b, q)$$

for, let us say, b (e.g., these are intensities to the left and right ears and the judgment is equal loudness), then b is really a random variable $\tilde{U}_{pq}(a)$. He then supposes that the appropriate additive representation is comparable to the analysis-of-variance models, namely,

$$\phi_A[\tilde{U}_{pq}(a)] = \phi_A(a) + \phi_P(p) - \phi_P(q) + \tilde{e}_{pq}(a),$$

where $\tilde{e}_{pq}(a)$ is a random variable with 0 median.

The key property used in the analysis is that if $\tilde{X}$ is a random variable, ϕ a strictly increasing function, and M the median operator, then

$$M[\phi(\tilde{X})] = \phi[M(\tilde{X})];$$

i.e., M and ϕ commute. Then, writing

$$m_{pq}(a) = M[\tilde{U}_{pq}(a)],$$

he proves from the representation that the following *cancellation* property must hold:

$$m_{pr}(a) = m_{pq}[m_{qr}(a)],$$

which corresponds to the double cancellation property. A second property, corresponding to transitivity, is *commutativity*:

$$m_{pq}[m_{rs}(a)] = m_{rs}[m_{pq}(a)].$$

Falmagne shows that if we define $\succsim$ on $A \times P$ by

$$(a, p) \succsim (b, q) \text{ iff } m_{pq}(a) \geqslant b$$

and assume that A and P are real intervals, that m is strictly increasing, and that m satisfies the cancellation and commutativity properties, then $\succsim$ is a weak order that satisfies double cancellation (and so independence because of solvability).

He then outlines methods of testing these two properties using median tests. So far, only pilot data for loudness summation have been published (Falmagne, 1976); they indicate small but systematic failures in the cancellation property, suggesting that additivity may not hold strictly. It is, however, much too early to be sure. Additional data will soon be reported.

6.7 CONCLUSIONS

It is reasonably clear that the simplest, best-known case of conjoint measurement, the additive representation, has limited direct application to human decision processes (section 6.5). Its main value is, first, in understanding more clearly the basic measurement structures of physics (sections 6.4.3 and 6.4.4), with the hope of eventually generalizing that structure to include behavioral and social science variables, and second, in providing a tool for analyzing the structure and representation of more complex conjoint structures. Examples of the latter are the study of certain nonadditive representations (section 6.3) and of conditional expected utility (section 6.4.2), which is of interest to decision analysts, economists, and statisticians. One can, therefore, anticipate considerable future work on the development and empirical testing of somewhat special, but still interesting, nonadditive representations. Experimental tests will continue to be somewhat frustrating until an adequate theory of error is evolved (section 6.6).

REFERENCES

Aczél, J. 1965. A remark on functional dependence. *Journal of Mathematical Psychology*, vol. 2, pp. 125–127.

Aczél, J., G. Pickert, and F. Radó. 1960. Nomogramme, Gewebe und Quasigruppen. *Mathematica*, Vol. 2, pp. 5–24.

Anderson, N.H. 1970. Functional measurement and psychophysical judgment. *Psychological Review*, Vol. 77, pp. 153–170.

Anderson, N.H. 1971. Integration theory and attitude change. *Psychological Review*, Vol. 78, pp. 171–206.

Anderson, N.H. 1973. Comments on the articles of Hodges and of Schönemann, Cafferty, and Rotton. *Psychological Review*, Vol. 80, pp. 88–92.

Anderson, N.H. 1974. Information integration theory: a brief survey. In *Contemporary Developments in Mathematical Psychology, II* (D.H. Krantz, R.C. Atkinson, R.D. Luce, and P. Suppes, eds.). Freeman, San Francisco, pp. 236–305.

Balch, M. 1974. On recent developments in subjective expected utility. In *Essays on Economic Behavior under Uncertainty* (M.S. Balch, D.L. McFadden, and S.Y. Wu, eds.). North-Holland, Amsterdam, pp. 45–55.

Balch, M., and P.C. Fishburn. 1974. Subjective expected utility for conditional primitives. In *Essays on Economic Behavior under Uncertainty* (M.S. Balch, D.L. McFadden, and S.Y. Wu, eds.). North-Holland, Amsterdam, pp. 57–69.

Becker, S.W., and F.O. Brownson. 1964. What price ambiguity? or the role of ambiguity in decision-making. *Journal of Political Economy*, Vol. 72, pp. 62–73.

Blaschke, W., and G. Bol. 1938. *Geometrie der Gewebe*. Springer, Berlin.

Campbell, B.A., and F.A. Masterson. 1969. Psychophysics of punishment. In *Punishment and Aversive Behavior*. (B.A. Campbell and R.M. Church, eds.). Appleton-Century-Crofts, New York, pp. 3–42.

Campbell, N.R. 1920. *Physics: the Elements*. Cambridge University Press, Cambridge. Reprinted in 1957 as *Foundations of Science: The Philosophy of Theory and Experiment*. Dover, New York.

Coombs, C.H., and J. Bowen. 1971. Additivity of risk in portfolios. *Perception and Psychophysics*, Vol. 10, pp. 43–46.

Coombs, C.H., and L.C. Huang. 1968. A portfolio theory of risk preference. Michigan Mathematical Psychology Program, Technical Report 68–5. University of Michigan, Ann Arbor.

Coombs, C.H., and L.C. Huang. 1970. Polynomial psychophysics of risk. *Journal of Mathematical Psychology*, Vol. 7, pp. 317–338.

Debreu, G. 1960. Topological methods in cardinal utility theory. In *Mathematical Methods in the Social Sciences, 1959* (K.J. Arrow, S. Karlin, and P. Suppes, eds.). Stanford University Press, Stanford, California.

Edwards, W. 1968. Conservatism in human information processing. In *Formal Representation of Human Judgment* (B. Kleinmunts, ed.). Wiley, New York.

Ellsberg, D. 1961. Risk, ambiguity, and the Savage axioms. *Quarterly Journal of Economics*, Vol. 75, pp. 643–669.

Falmagne, J.C. 1973. Sufficient conditions for an ordered-ring isomorphism onto a positive subinterval: a lemma for polynomical measurement. *Journal of Mathematical Psychology*, Vol. 10, pp. 290–295.

Falmagne, J.C. 1976. Random conjoint measurement and loudness summation. *Psychological Review*, Vol. 83, pp. 65–79.

Feldman, J., and J.C. Baird. 1971. Magnitude estimation of multidimensional stimuli. *Perception and Psychophysics*, Vol. 10, pp. 418–422.

Fishburn, P.C. 1974. On the foundations of decision making under uncertainty. In *Essays on Economic Behavior under Uncertainty* (M.S. Balch, D.L. McFadden, and S.Y. Wu, eds.). North-Holland, Amsterdam.

Fishburn, P.C. 1975. Nondecomposable conjoint measurement for bisymmetric structures. *Journal of Mathematical Psychology*, Vol. 12, pp. 75–89.

Green, D.M., and R.D. Luce. 1974. Variability of magnitude estimates: a timing theory analysis. *Perception and Psychophysics*, Vol. 15, pp. 291–300.

Green, P.E. 1974. On the design of choice experiments involving multifactor alternatives. *Journal of Consumer Research*, Vol. 1, pp. 61–68.

Green, P.E., and M.T. Devita. 1974. A complementarity model of consumer utility for item collections. *Journal of Consumer Research*, Vol. 1, pp. 56–67.

Green, P.E., and V.R. Rao. 1971. Conjoint measurement for quantifying judgmental data. *Journal of Marketing Research*, Vol. 8, pp. 355–363.

Green, P.E., and Y. Wind. 1975. New Way to measure consumers' judgments. *Harvard Business Review*, pp. 107–117.

Havel, V. 1966. Verallgemeinerte Gewebe. I. *Arch. Math. (Brno)*, Vol. 2, pp. 63–70.

Hodges, B.H. 1973. Adding and averaging models for information integration. *Psychological Review*, Vol. 80, pp. 80–84.

Holman, E.W. 1971. A note on additive conjoint measurement. *Journal of Mathematical Psychology*, Vol. 8, pp. 489–494.

Hull, C.L. 1952. *A Behavior System*. Yale University Press, New Haven, Connecticut.

Jaffray, J.Y. 1974. On the extension of additive utilities to infinite sets. *Journal of Mathematical Psychology*, Vol. 11, pp. 431–452.

Krantz, D.H. 1968. A survey of measurement theory. In *Mathematics of the Decision Sciences, Part 2* (G.B. Dantzig and A.F. Veinott, Jr., eds.). Vol. 12. Lectures in Applied Mathematics, American Mathematical Society, Providence, Rhode Island, pp. 314–350.

Krantz, D.H. 1972. Measurement structures and psychological laws. *Science*, Vol. 175, pp. 1427–1435.

Krantz, D.H. 1974. Measurement theory and qualitative laws in psychophysics. In *Contemporary Developments in Mathematical Psychology, II* (D.H. Krantz, R.C. Atkinson, R.D. Luce, and P. Suppes, eds.). Freeman, San Francisco, pp. 160–199.

Krantz, D.H., and R.D. Luce. 1974. The interpretation of conditional expected-utility theories. In *Essays on Economic Behavior under Uncertainty* (M.S. Balch, D.L. McFadden, and S.Y. Wu, eds.). North-Holland, Amsterdam, pp. 70–73.

Krantz, D.H., R.D. Luce, P. Suppes, and A. Tversky. 1971. *Foundations of Measurement, I.* Academic Press, New York.

Krantz, D.H., and A. Tversky. 1971. Conjoint-measurement analysis of composition rules in psychology. *Psychological Review*, Vol. 78, pp. 151–169.

Levelt, W.J.M., J.B. Riemersa, and A.A. Bunt. 1972. Binaural additivity of loudness. *British Journal of Mathematical and Statistical Psychology*, Vol. 25, pp. 51–68.

Levine, M.V. 1970. Transformations that render curves parallel. *Journal of Mathematical Psychology*, Vol. 7, pp. 410–444.

Levine, M.V. 1972. Transforming curves into curves with the same shape. *Journal of Mathematical Psychology*, Vol. 9, pp. 1–16.

Levine, M.V. 1975a. Additive measurement with short segments of curves. *Journal of Mathematical Psychology*, Vol. 12, pp. 212–224.

Levine, M.V. 1975b. Appropriateness measurement and test bias. Unpublished manuscript.

Luce, R.D. 1965. A "fundamental" axiomatization of multiplicative power relations among three variables. *Philosophy of Science*, Vol. 32, pp. 301–309.

Luce, R.D. 1966. Two extensions of conjoint measurement. *Journal of Mathematical Psychology*, Vol. 3, pp. 348–370.

Luce, R.D. 1976. Dimensionally invariant numerical laws correspond to meaningful relations. Unpublished manuscript.

Luce, R.D., and D.H. Krantz. 1971. Conditional expected utility. *Econometrica*, Vol. 39, pp. 253–271.

Luce, R.D., and L. Narens. In press. A qualitative equivalent to the relativistic addition law for velocities. *Synthese*.

Luce, R.D., and J. Tukey. 1964. Simultaneous conjoint measurement: a new type of fundamental measurement. *Journal of Mathematical Psychology*, Vol. 1, pp. 1–27.

MacCrimmon, K.R. 1968. Descriptive and normative implications of the decision-theory postulates. In *Risk and Uncertainty* (K. Borch and J. Mossin, eds.). Macmillan, New York, pp. 3–32.

MacCrimmon, K.R., and S. Larsson. In press. Utility theory: axioms versus "paradoxes." *Theory and Decision*.

Marks, L.E. 1974. *Sensory Processes, the New Psychophysics*. Academic Press, New York.

Moskowitz, H.R., B. Scharf, and J.C. Stevens, eds. 1974. *Sensation and Measurement*. Reidel, Dordrecht.

Mosteller, F., and P. Nogee. 1951. An experimental measurement of utility. *Journal of Political Economy*, Vol. 59, pp. 371–404.

Nagel, E. 1961. *The Structure of Science*. Harcourt, Brace & World, New York.

Narens, L. 1974a. Measurement without Archimedean axioms. *Philosophy of Science*, Vol. 41, pp. 374–393.

Narens, L. 1974b. Minimal conditions for additive conjoint measurement and qualitative probability. *Journal of Mathematical Psychology*, Vol. 11, pp. 404–430.

Narens, L., and R.D. Luce. 1975. The algebra of measurement. Unpublished manuscript.

Payne, J.W. 1973a. Alternative approaches to decision making under risk: Moments versus risk dimensions. *Psychological Bulletin*, Vol. 80, pp. 439–454.

Payne, J.W. 1973b. Contingent information-processing in decision making under risk: the role of the basic risk dimensions. Ph.D. dissertion. University of California, Irvine.

Payne, J.W., and M.L. Braunstein. 1971. Preferences among gambles with equal underlying distributions. *Journal of Experimental Psychology*, Vol. 87, pp. 13–18.

Pfanzagl, J. 1971. *Theory of Measurement*. 2nd ed. Physica-Verlag, Vienna.

Phillips, L.D., and W. Edwards. 1966. Conservatism in a simple probability inference task. *Journal of Experimental Psychology*, Vol. 72, pp. 346–354.

Pollatsek, A., and A. Tversky. 1970. A theory of risk. *Journal of Mathematical Psychology*, Vol. 7, pp. 540–553.

Radó, F. 1969. Eine Bedingung fur die Regularität der Gewebe. *Mathematica (Cluj)*, Vol. 2, pp. 325–334.

Radó, F. 1965. Einbettung eines Halbgewebes in ein reguläres Gewebe und eines Halbgruppoids in eine Gruppe. *Mathematische Zeitschrift*, Vol. 89, pp. 395–410.

Ramsey, J.O. 1975. Review of Krantz *et al. Foundations of Measurement, Vol. 1. Psychometrika*, Vol. 40, pp. 257–262.

Richter, M.K. 1975. Rational choice and polynomial measurement models. *Journal of Mathematical Psychology*, Vol. 12, pp. 99–113.

Savage, L.J. 1954. *The Foundations of Statistics*. John Wiley, New York.

Scheffé, H. 1959. *The Analysis of Variance*. John Wiley, New York.

Schönemann, P.H., T. Cafferty, and J. Rotton. 1973. A note on additive functional measurement. *Psychological Review*, Vol. 80, pp. 85–87.

Scott, D. 1964. Measurement structures and linear inequalities. *Journal of Mathematical Psychology*, Vol. 1, pp. 233–247.

Scott, D., and P. Suppes. 1958. Foundational aspects of theories of measurement. *Journal of Symbolic Logic*, Vol. 23, pp. 113–128.

Skala, H.J. 1975. *Non-Archimedean Utility Theory.* Reidel, Boston.

Slovic, P. 1967. The relative influence of probabilities and payoffs upon perceived risk of a gamble. *Psychonomic Science*, Vol. 1, pp. 223–224.

Slovic, P., and S. Lichtenstein. 1968. Relative importance of probabilities and payoffs in risk taking. *Journal of Experimental Psychology Monograph*, Vol. 78, pp. 1–18.

Spence, K.W. 1956. *Behavior Theory and Conditioning.* Yale University Press, New Haven, Connecticut.

Stevens, S.S. 1957. On the psychophysical law. *Psychological Review*, Vol. 64, pp. 153–181.

Stevens, S.S. 1971. Issues in psychophysical measurement. *Psychological Review*, Vol. 78, pp. 426–450.

Stevens, S.S. 1975. *Psychophysics.* John Wiley, New York.

Tversky, A. 1964. Finite additive structures. Michigan Mathematical Psychology Program, Technical Report MMPP 64-6. University of Michigan, Ann Arbor.

Tversky, A. 1967a. A general theory of conjoint measurement. *Journal of Mathematical Psychology*, Vol. 4, pp. 1–20.

Tversky, A. 1967b. Additivity, utility, and subjective probability. *Journal of Mathematical Psychology*, Vol. 4, pp. 175–201.

Tversky, A., and D.H. Krantz. 1969. Similarity of schematic faces: a test of interdimensional additivity. *Perception and Psychophysics*, Vol. 5, pp. 124–128.

Tversky, A., and A. Zivian. 1966. A computer program for additivity analysis. *Behavioral Science*, Vol. 11, pp. 78–79.

Wallsten, T.S. 1972. Conjoint-measurement framework for the study of probabilistic information processing. *Psychological Review*, Vol. 79, pp. 245–260.

Wallsten, T.S. In press (a). Using conjoint measurement models to investigate a theory about probabilistic information processing. *Journal of Mathematical Psychology*, Vol. 13.

Wallsten, T.S. In press (b). Processing information for decisions. In *Cognitive Theory* (N.J. Castellan, D. Pisoni, and G. Potts, eds.). Vol. 2. Erlbaum Associates, New York.

Wallsten, T.S., and M.M. Sapp. In press. Strong ordinal properties of additive models for sequential processing of probabilistic information. *Acta Psychologica*, Vol. 44.

Young, F. 1973. Conjoint scaling. The L.L. Thurstone Psychometric Laboratory Technical Report 118. The University of North Carolina, Chapel Hill.